Random model for the moments of the eigenfunctions of a point-scatterer

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joint work with H. Ueberschär

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Why point-scatterers?

Classical vs quantum dynamics

Point mass in some ambient space (M, g).

Classical

- Phase space T^*M (cotangent bundle).
- Dynamics governed by a differential equation.
- Want to understand its flow.

Example: geodesic flow on (M, g).

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Example: Laplacian Δ.

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Semi-classical analysis

Relate the classical dynamics to the properties of the eigenfunctions, in the semi-classical limit: ϕ_λ eigenfunction with eigenvalue λ and $\lambda \to +\infty$.

Model systems

Integrable case

Model system: 2-dimensional flat torus.

- Classical: explicit geodesic flow.
- Quantum: reasonnably explicit spectrum and eigenfunctions.

Chaotic case

Model system: hyperbolic surface

- Classical: explicit geodesic flow.
- Quantum: non-explicit spectrum and eigenfunctions.

Point-scatterer (informal version)

A point-scatterer on M at x is an operator that can be thought of as "$\Delta + \delta_x$", where

$$(\Delta + \delta_x)f = \Delta f + f(x)\delta_x.$$

Quantum version of the geodesic flow on M with a point obstacle at x.

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On a flat torus (Šeba billiard):

- classical dynamics still integrable;
- quantum system exhibit many features of quantum chaos;
- reasonnably explicit spectrum and eigenfunctions.

Berry's Conjecture

(M, g) with chaotic geodesic flow. X uniform random variable on M.
ϕ_λ Laplace eigenfunction associated with λ.

Weak Berry's Conjecture

The random variable $\phi_\lambda(X)$ satisfies a Central Limit Theorem as $\lambda \to +\infty$:

$$\frac{\phi_\lambda(X) - \mathbb{E}[\phi_\lambda(X)]}{\sqrt{\mathrm{Var}(\phi_\lambda(X))}} \xrightarrow[\lambda \to +\infty]{\text{distribution}} \mathcal{N}(0, 1).$$

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Moments Conjecture

For all $p \in \mathbb{N}^*$, $\mathbb{E}\left[\left(\frac{\phi_\lambda(X) - \mathbb{E}[\phi_\lambda(X)]}{\sqrt{\mathrm{Var}(\phi_\lambda(X))}}\right)^p\right] \xrightarrow[\lambda\to+\infty]{} \mu_p$, where

$$\mu_p = \mathbb{E}[\mathcal{N}(0,1)^p] = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ (p-1)(p-3)\cdots 1 & \text{if } p \text{ is even.} \end{cases}$$

New eigenfunctions of a point-scatterer on a torus

Point-scatterers on rectangular flat tori

Ambient space

- $\mathbb{T}_\alpha = \mathbb{R}^2 / \left(\alpha\mathbb{Z} \oplus \frac{1}{\alpha}\mathbb{Z}\right)$ with $\alpha > 0$;
- dx Lebesgue measure, such that $\mathrm{Vol}\,(\mathbb{T}_\alpha) = 1$.

Laplacian $\Delta = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$: self-adjoint positive operator on $L^2(\mathbb{T}_\alpha)$.

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Theorem (von Neumann)

Denoting by $D_0 = \mathcal{C}_c^\infty(\mathbb{T}_\alpha \setminus \{0\})$, there exists a one-parameter family $(\Delta_\varphi)_{\varphi\in(-\pi,\pi]}$ of self-adjoint extensions of $\Delta_{|D_0}$ to $L^2(\mathbb{T}_\alpha)$.

If $\varphi = \pi$ we recover Δ, else we say that Δ_φ is a *point-scatterer*.

Laplace spectrum on $\mathbb{T}_\alpha$

$$\mathrm{Sp}(\Delta) = \left\{ 4\pi^2 \left(\frac{a^2}{\alpha^2} + \alpha^2 b^2 \right) \,\middle|\, a, b \in \mathbb{N} \right\} = \{\lambda_k \mid k \geqslant 0\},$$

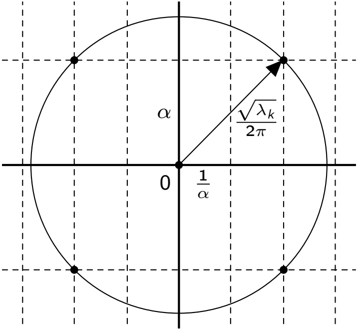
where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots \xrightarrow[k \to +\infty]{} +\infty$.

$$\Lambda_k \;=\; \left\{ \xi \in \tfrac{1}{\alpha}\mathbb{Z} \oplus \alpha\mathbb{Z} \,\middle|\, \|\xi\| = \tfrac{\sqrt{\lambda_k}}{2\pi} \right\}$$

wave vectors associated with λ_k.

$\left\{ e^{2i\pi\langle \xi\, ,\cdot \rangle} \,\middle|\, \xi \in \Lambda_k \right\}$ orthonormal basis of $\ker(\Delta - \lambda_k \,\mathrm{Id}) \subset L^2(\mathbb{T}_\alpha)$.

$r_k = \mathrm{card}(\Lambda_k)$ *multiplicity* of λ_k.



Weyl Law

Spectrum counting function

$$N(\lambda) = \sum_{\lambda_k \leqslant \lambda} r_k = \text{card}\left\{ \xi \in \frac{1}{\alpha}\mathbb{Z} \oplus \alpha\mathbb{Z} \;\middle|\; \|\xi\| \leqslant \frac{\sqrt{\lambda}}{2\pi} \right\}.$$

Weyl Law

We have $N(\lambda) = \frac{\lambda}{4\pi} + O\left(\sqrt{\lambda}\right)$ as $\lambda \to +\infty$.

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Weyl Law

We have $N(\lambda) = \frac{\lambda}{4\pi} + O\left(\sqrt{\lambda}\right)$ as $\lambda \to +\infty$.

- If $\alpha^4 \notin \mathbb{Q}$ (irrational tori), $r_k \in \{1, 2, 4\}$ and generically $r_k = 4$.
- If $\alpha = 1$ (square torus), $r_k = 8$ infinitely many times (density 0 subsequence). Besides, as $n \to +\infty$,

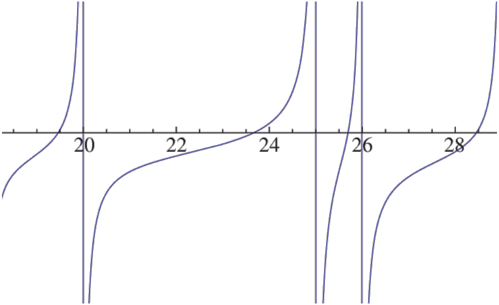
$$\frac{1}{n+1}\sum_{k=0}^{n} r_k \sim C\sqrt{\ln(\lambda_n)}.$$

Spectrum of the point-scatterer Δ_φ ($\varphi \neq \pi$)

We have $\mathrm{Sp}(\Delta_\varphi) = \{\lambda_k \mid k \geqslant 1\} \sqcup \left\{\tau_k^\varphi \mid k \geqslant 0\right\}$.

- λ_k of multiplicity $r_k - 1$, associated with $\{\phi \in \ker(\Delta - \lambda_k \mathrm{Id}) \mid \phi(0) = 0\}$.
- τ_k^φ simple eigenvalue. It's the $(k+1)$-th solution of:

$$\sum_{k \geqslant 0} r_k \left(\frac{1}{\lambda_k - \tau} - \frac{\lambda_k}{\lambda_k^2 + 1} \right) = \tan\left(\frac{\varphi}{2}\right) \sum_{k \geqslant 0} \frac{r_k}{\lambda_k^2 + 1}.$$



New eigenfunctions of Δ_φ ($\varphi \neq \pi$)

Let $\tau \in \mathbb{R} \setminus \mathrm{Sp}(\Delta)$, we denote $G_\tau = -\frac{1}{\tau} + \sum_{k \geqslant 1} \sum_{\xi \in \Lambda_k} \frac{e^{2i\pi\langle \xi , \cdot \rangle}}{\lambda_k - \tau}$ from $\mathbb{T}_\alpha$ to $\mathbb{R}$.

New eigenfunctions

If τ is one of the new eigenvalues, then $(\Delta_\varphi - \tau \, \mathrm{Id}) G_\tau = 0$.

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Moments of G_τ

Let $p \in \mathbb{N}^*$ and $\tau \in \mathbb{R} \setminus \mathrm{Sp}(\Delta)$, the p-th central *moment* of G_τ is:

$$M_\tau^p = \int_{\mathbb{T}_\alpha} \left(G_\tau(x) + \frac{1}{\tau} \right)^p \mathrm{d}x.$$

We have $M_\tau^1 = 0$ and $M_\tau^2 = \sum_{k \geqslant 1} \frac{r_k}{(\lambda_k - \tau)^2}$.

Deterministic problem

Question

Do we have $\dfrac{M_\tau^p}{(M_\tau^2)^{\frac{p}{2}}} \xrightarrow[\tau \to +\infty]{} \mu_p$ for any $p \geqslant 3$?

- Conjectured by Šeba (1990).
- Keating–Marklov–Winn (2003) argue that it's not always true.
- Kurlberg–Ueberschär (2019): if α^4 is diophantine then

$$\frac{M_\tau^4}{(M_\tau^2)^2} \nrightarrow \mu_4,$$

not even along sequences of the form $(\tau_k^\varphi)_{k \geqslant 0}$ with $\varphi \neq \pi$.

The Berry–Tabor Conjecture

Poisson point processes

A random variable N in $\mathbb{N}$ is Poisson distributed with parameter $\nu \geqslant 0$ if $\mathbb{P}(N = k) = e^{-\nu}\frac{\nu^k}{k!}$ for all $k \in \mathbb{N}$. We denote this by $N \sim \propto(\nu)$.

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Let ν be a measure on $[0, +\infty)$, a *Poisson point process* with intensity ν is a random subset $P \subset [0, +\infty)$ such that:

- for any Borel subset B, $\mathrm{card}(P \cap B) \sim \propto(\nu(B))$.
- for any disjoint Borel subsets $B_1, \ldots, B_n$, the random variables $(\mathrm{card}(P \cap B_i))_{1 \leqslant i \leqslant n}$ are independent.

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If $\nu([0, +\infty)) = +\infty$, then almost surely the elements of P can be ordered into a sequence $(\lambda_k)_{k \geqslant 1}$ such that:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \xrightarrow[k \to +\infty]{} +\infty.$$

The Berry–Tabor Conjecture

Conjecture (Berry–Tabor)

On $\mathbb{T}_\alpha$ the sequence $(\lambda_k)_{k \geqslant 1}$ of positive eigenvalues of Δ "behaves like" a Poisson point process.

If $\alpha^4 \notin \mathbb{Q}$, generically $r_k = 4$. In order to agree with Weyl's Law, we need:

$$4\nu([0,\lambda]) = 4\mathbb{E}[\mathrm{card}(P \cap [0,\lambda])] = N(\lambda) \sim \frac{\lambda}{4\pi},$$

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- Numerics for $\alpha^4 \notin \mathbb{Q}$: $\displaystyle \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_k - \lambda_{k-1}} \xrightarrow[N\to+\infty]{\text{distribution}} \mathcal{E}\left(\frac{1}{16\pi}\right)$.
- Sarnak: $\displaystyle \frac{1}{N}\sum_{1\leqslant k,l\leqslant N} \delta_{\lambda_k - \lambda_l}$ admits a Poissonian limit for a.e. flat torus.

A simple plan

G_τ only depends on τ and the sequences $(\lambda_k)_{k\geqslant 1}$ and $(\Lambda_k)_{k\geqslant 1}$.

- Replace $\mathrm{Sp}(\Delta)$ with a Poisson point process.
- Tune its intensity in order to agree with Weyl's Law.
- Choose directions in $[0, \frac{\pi}{2}]$ for the wave vectors in $\Lambda_k \cap [0, +\infty)^2$ and take the closure under symmetry with respect to the coordinate axes (for example: independent uniform directions).

A too simple plan

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Problems

- G_τ no longer defines a function on $\mathbb{T}_\alpha$.
- Interactions between ν and the multiplicities $(r_k)_{k\geqslant 1}$.

Definition of the random model

Step 1: deterministic expression of the moments

Given $a = (a_k)_{k \geqslant 1}$ with values in $\mathbb{N}$ and finite support, we denote:

- $|a| = \sum_{k \geqslant 1} a_k$,
- $a! = \prod_{k \geqslant 1} a_k!$,
- $$N_a = \operatorname{card} \left\{ (\xi_{k,l})_{1 \leqslant l \leqslant a_k} \in \prod_{k \geqslant 1} (\Lambda_k)^{a_k} \;\middle|\; \sum_{k \geqslant 1} \sum_{l=1}^{a_k} \xi_{k,l} = 0 \right\}.$$

Lemma

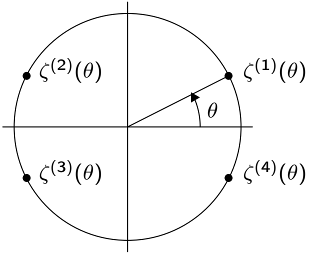
For all $p \geqslant 1$ and $\tau \in \mathbb{R} \setminus \mathrm{Sp}(\Delta)$, we have:

$$M_\tau^p = p! \sum_{|a|=p} \frac{N_a}{a!} \prod_{k \geqslant 1} \left(\frac{1}{\lambda_k - \tau} \right)^{a_k}.$$

Step 2: randomization of the wave vectors

If $\theta \in [0, \frac{\pi}{2}]$, $\zeta^{(1)}(\theta) = (\cos(\theta), \sin(\theta))$.

η measure on $[0, \frac{\pi}{2}]^{\mathbb{N}^* \times \mathbb{N}^*}$, distribution of a sequence of independent uniform random variables.



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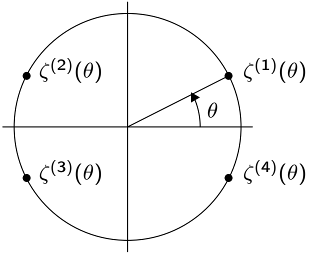

Randomized wave vectors

We choose:

- $(\lambda_k)_{k \geqslant 1}$ increasing sequence of positive numbers;
- $(m_k)_{k \geqslant 1}$ sequence with values in $\mathbb{N}^*$;
- $(\theta_{k,l})_{k,l \geqslant 1}$ random variables in $[0, \frac{\pi}{2}]$, with a density with respect to η.

We redefine $\Lambda_k = \frac{\sqrt{\lambda_k}}{2\pi} \cdot \left\{ \zeta^{(i)}(\theta_{k,j}) \,\middle|\, 1 \leqslant i \leqslant 4 \text{ et } 1 \leqslant j \leqslant m_k \right\}$.

Spectral sums and almost sure expression

Spectral sums

Let $q \geqslant 1$ and $\tau \in \mathbb{R}$, we set $S_\tau^q = \sum_{k \geqslant 1} \frac{m_k}{(\lambda_k - \tau)^{2q}} \in [0, +\infty]$.

Proposition

Almost surely, for any $p \geqslant 1$ and $\tau \in \mathbb{R} \setminus \{\lambda_k \mid k \geqslant 0\}$ we have:

- $M_\tau^{2p-1} = 0$;
- *if $S_\tau^q < +\infty$ for all $q \in \{1, \dots p\}$ then $M_\tau^{2p} = P_p(S_\tau^1, S_\tau^2, \dots, S_\tau^p)$, where $P_p \in \mathbb{R}[X_1, \dots, X_p]$ is deterministic, explicit, depends only on p.*

Almost surely, $\mathrm{card}(\Lambda_k) = 4m_k$ for all $k \geqslant 1$.

Step 3: randomization of the Laplace spectrum

Randomized eigenvalues and multiplicities

We choose $m : [0, +\infty) \to [1, +\infty)$.

- We model $(\lambda_k)_{k \geqslant 1}$ by a Poisson process with intensity $\nu_m = \dfrac{1}{16\pi m} \, \mathrm{d}t$.
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We say that m is a *multiplicity function* if:

- m is $\mathcal{C}^1$,
- there exists $\beta > 0$ such that $m'(t) = O(t^{-\beta})$ as $t \to +\infty$.

Examples

- $m : t \mapsto 1$ (irrational tori).
- $m : t \mapsto 1 + C\sqrt{\ln(1+t)}$ (average behavior on the square torus).

Results for the random model

Main result (L.-Ueberschär, 2019)

Let $p \geqslant 1$ and $\tau \in \mathbb{R}$, the randomized moment $M_\tau^{2p} = P_p(S_\tau^1, \ldots, S_\tau^p)$ is almost surely well-defined.

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Let $p \geqslant 1$ and $\tau \in \mathbb{R}$, the randomized moment $M_\tau^{2p} = P_p(S_\tau^1, \ldots, S_\tau^p)$ is almost surely well-defined.

Given $p \geqslant 2$, there exists a one-parameter family $(R_p(\ell))_{0 \leqslant \ell \leqslant +\infty}$ of random variables such that: if $m(\tau) \xrightarrow[\tau \to +\infty]{} \ell$ then

$$\frac{M_\tau^{2p}}{(M_\tau^2)^p} \xrightarrow[\tau \to +\infty]{\text{distribution}} \mu_{2p} R_p(\ell).$$

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- The distribution of $R_p(\ell)$ only depends on ℓ.
- $R_p(\ell) = R_p(\ell')$ in distribution if and only if $\ell = \ell'$.
- If $\ell < +\infty$, then $R_p(\ell)$ admits a smooth density.
- If $\ell = +\infty$, then $R_p(\ell) = 1$ a.s. and convergence holds in probability.

The end

Thank you for your attention.